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On the Asymptotic Behavior of Random Walks on an Anisotropic Lattice

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A random walk on a two-dimensional lattice with homogeneous rows and inhomogeneous columns, which could serve as a model for the study of some transport phemonema, is discussed. Subject to an asymptotic density condition on the columns it is shown that the horizontal motion of the walk is asymptotically like that of rescaled Brownian motion. Various consequences of this are derived including central limit, iterated logarithm, and mean square displacement results for the horizontal component of the walk.

KEY WORDS: Random walks; anisotropic lattices; Brownian motion; martingale methods; central limit theorem; law of the iterated logarithm.

1. INTRODUCTION

A number of recent papers, in particular Refs. 1 and 2, have discussed random walks on certain anistropic lattices which could serve as a model for the study of transport phenomena. These lattices are two dimensional with homogeneous rows but inhomogeneous columns. In Ref. 1 the case of two types of columns was considered and in Ref. 2 any number of column types is allowed but a certain asymptotic density is required to exist uniformly over the array of types of columns. The principal object of study for such a general array was the mean square displacements of the horizontal and vertical components of the walk.

In this paper we shall adopt a more general approach to the problem and will show that the horizontal component of the walk behaves asymptotically like a rescaled Brownian motion. Detailed limiting behavior can then readily be obtained, and in particular the mean square displacement result of Westcott[(2)] but under a somewhat different asymptotic density condition for the columns.

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2. THE MODEL AND THE PRINCIPAL RESULTS

We shall consider a random walk which, if situated at a site on column j, moves with probability p_j to either horizontal neighbor and with probability $\frac{1}{2} - p_j$ to either vertical neighbor at the next step. Thus, at any time the transition mechanism depends only on the index of the column which is at present occupied.

Let X_n denote the horizontal position of the walk after n steps ($X_0 = 0$) so that the transition probabilities are

$$P(X_{n+1} = j \pm 1 \mid X_n = j) = p_j$$
$$P(X_{n+1} = j \mid X_n = j) = 1 - 2p_j$$

for each $j \in \mathbb{Z}$, the set of integers, and $n = 0, 1, 2, \ldots$. We shall suppose that the $\{p_j, j \in \mathbb{Z}\}$ satisfy the asymptotic density condition

$$k^{-1} \sum_{j=1}^{k} p_j^{-1} = 2\gamma + o(k^{-\eta}), \qquad k^{-1} \sum_{j=1}^{k} p_{-j} = 2\gamma + o(k^{-\eta}) \tag{1}$$

as $k \to \infty$ for some constants γ, $\frac{1}{2} < \gamma < \infty$ and $\eta > \frac{1}{2}$.

Note that the condition of Eq. (1) is satisfied for any strictly periodic array of columns, for which there exists a positive integer Q such that

$$p_j = p_{j+Q}, \qquad j \in \mathbb{Z}$$

but neither implies nor is implied by the uniformity condition

$$\lim_{k\to\infty} \sup_{r \in \mathbb{Z}} \left| k^{-1} \sum_{j=r}^{r+k} p_j^{-1} - 2\gamma \right| = 0$$

of Westcott.[2]

In the results below we shall use the notation $\log_2 n$ for $\log\log n$, a.s. for almost surely (i.e., with probability one), $\xrightarrow{d}$ for convergence in distribution, and $N(0,1)$ for the unit normal law.

We shall establish the following results.

Theorem 1. The probability space on which $\{X_n, n \geqslant 0\}$ is defined may be expanded so as to carry a standard Brownian motion $\{W(t), t \geqslant 0\}$ such that

$$\gamma^{1/2} X_n = W(n(1+\epsilon_n)) + O\left(n^{1/4}(\log n)^{1/2}(\log_2 n)^{1/2}\right) \text{ a.s.} \tag{2}$$

as $n \to \infty$ where $\{\epsilon_n\}$ satisfies $\lim_{n\to\infty} \epsilon_n = 0$ a.s.

Theorem 1 shows that the asymptotic behavior of $\gamma^{1/2} X_n$ is that of the randomly rescaled Brownian motion $W(n(1+\epsilon_n))$. The error of the approximation should be viewed in the light of the law of the iterated logarithm for Brownian motion which gives $\limsup_{n\to\infty}|W(n)| = O((n\log_2 n)^{1/2})$ a.s.

If strong additional assumptions are made about the column types, for example if there is a finite number of different types which occur with fixed periodicity, then the random variable ϵ_n can safely be removed and relegated to the error in (2). We shall not pursue this question here because the presence of ϵ_n does not cause undue complication in the extraction of specific results from (2) as evidenced by Corollary 1 below.

Corollary 1.

(i) $$\gamma^{1/2}n^{-1/2}X_n \xrightarrow{d} N(0,1) \qquad \text{as} \quad n\to\infty$$

(ii) $$\overline{\lim_{n\to\infty}}\,(2n\gamma^{-1}\log_2 n)^{-1/2}X_n = 1 \text{ a.s.},$$

$$\underline{\lim_{n\to\infty}}\,(2n\gamma^{-1}\log_2 n)^{-1/2}X_n = -1 \text{ a.s.}$$

Corollary 2.

$$\langle X_n^2\rangle \sim \gamma^{-1}n \qquad \text{as} \quad n\to\infty.$$

An anonymous referee of the paper[2] has shown that (i) of Corollary 1 can be obtained under the weaker condition $\eta = 0$.

3. PROOFS

Let $\sigma_0 = 0 < \sigma_1 < \sigma_2 < \cdots$ be the successive times at which the values of the $X_i - X_{i-1}$, $i = 1,2,\ldots$ are nonzero and put $S_k = X_{\sigma_k}$. By the assumed symmetry, $\{S_k,\ k \geq 0\}$ is a simple random walk. We shall write $S_k = \sum_{j=1}^k Y_j$, $k \geq 1$, the Y_j being independent and identically distributed with $P(Y_j = -1) = P(Y_j = 1) = \frac{1}{2}$.

The principal result which we need in order to establish Theorem 1 is given in Theorem 2 below. However, as a prelude we require the following lemma.

Lemma 1. $\sum_1^\infty n^{-2}\langle p_{S_n}^{-2}\rangle < \infty$ and hence $\sum_1^\infty n^{-2}p_{S_n}^{-2} < \infty$ a.s.

Proof. From Eq. (1) we find that $p_j^{-1} = o(|j|^{1-\eta})$, $\frac{1}{2} < \eta < 1$ as $|j|\to\infty$. This ensures the existence of $C > 0$ for which $p_j^{-1} \leq C|j|^{1-\eta}$, all $j \neq 0$. Then,

$$\sum_1^\infty n^{-2}\langle p_{S_n}^{-2}\rangle = \sum_{\{n:S_n=0\}} n^{-2}\langle p_{S_n}^{-2}\rangle + \sum_{\{n:S_n\neq 0\}} n^{-2}\langle p_{S_n}^{-2}\rangle$$

and

$$\sum_{\{n:S_n=0\}} n^{-2}\left\langle p_{S_n}^{-2}\right\rangle \leqslant p_0^{-2}\sum_1^{\infty} n^{-2} < \infty$$

while

$$\sum_{\{n:S_n\neq 0\}} n^{-2}\left\langle p_{S_n}^{-2}\right\rangle \leqslant C\sum_1^{\infty} n^{-2}\langle |S_n|\rangle^{2(1-\eta)} < \infty$$

since

$$\langle |S_n|\rangle^{2(1-\eta)} \leqslant \left\langle S_n^2\right\rangle^{1-\eta} = n^{1-\eta}$$

using Liapounov's inequality. This, together with an application of the Borel–Cantelli lemmas, completes the proof. ■

Now we are in a position to establish the following result which is of independent interest.

Theorem 2.

$$n^{-1}\sigma_n \overset{\text{a.s.}}{\to} \gamma \qquad \text{as} \quad n \to \infty.$$

Proof. Writing $\mathfrak{F}_k$ for the σ-field generated by $S_1, \ldots, S_k$, $\sigma_1, \ldots, \sigma_k$, $k \geqslant 1$, and $\mathfrak{F}_0$ for the trivial σ-field, we have

$$P(\sigma_{j+1} = \sigma_j + r \mid \mathfrak{F}_j) = \left(1 - 2p_{S_j}\right)^{r-1} 2p_{S_j}, \qquad r = 1, 2, \ldots$$

and hence

$$\left\langle \sigma_{j+1} - \sigma_j \mid \mathfrak{F}_j\right\rangle = \left(2p_{S_j}\right)^{-1} \tag{3}$$

$$\begin{aligned}\left\langle (\sigma_{j+1} - \sigma_j)^2 \mid \mathfrak{F}_j\right\rangle &= \left(2p_{S_j}\right)^{-1}\left(p_{S_j}^{-1} - 1\right)\\ &\leqslant \left(2p_{S_j}^2\right)^{-1}\end{aligned} \tag{4}$$

Using Eq. (3) we have that $(W_n, \mathfrak{F}_n, n \geqslant 0)$ defined by

$$W_n = \sum_{j=1}^{n}\left[\sigma_j - \langle \sigma_j \mid \mathfrak{F}_{j-1}\rangle\right] = \sigma_n - \frac{1}{2}\sum_{j=0}^{n-1} p_{S_j}^{-1} \tag{5}$$

is a martingale. Furthermore, using Eqs. (3) and (4),

$$\left\langle (\sigma_{j+1} - \langle \sigma_{j+1} \mid \mathfrak{F}_j\rangle)^2 \mid \mathfrak{F}_j\right\rangle = \left\langle \left(\sigma_{j+1} - \sigma_j - \tfrac{1}{2} p_{S_j}^{-1}\right)^2 \mid \mathfrak{F}_j\right\rangle \leqslant \tfrac{1}{4} p_{S_j}^{-2}$$

so that

$$\sum_{j=1}^{\infty} j^{-2}\left\langle (\sigma_j - \langle \sigma_j \mid \mathfrak{F}_{j-1}\rangle)^2\right\rangle < \infty$$

by virtue of Lemma 1. A standard martingale strong law (e.g., Ref. 3, Theorem 2.18, p. 35) then gives

$$n^{-1}\sum_{j=1}^{n}(\sigma_j - \langle\sigma_j \mid \mathcal{F}_{j-1}\rangle) \overset{\text{a.s.}}{\to} 0$$

and hence, in the light of Eq. (5), it suffices to show that

$$n^{-1}\sum_{j=0}^{n-1} p_{S_j}^{-1} \overset{\text{a.s.}}{\to} 2\gamma \tag{6}$$

as $n \to \infty$ in order to obtain the result of the theorem.

Now set $N_n(k)$ for the number of S_j, $0 \leqslant j \leqslant n$, for which $S_j = k$, and write

$$\alpha_k = k^{-1}\sum_{j=1}^{k} p_j^{-1}, \qquad \beta_k = k^{-1}\sum_{j=1}^{k} p_{-j}^{-1}$$

recalling that $\alpha_k \to 2\gamma$, $\beta_k \to 2\gamma$ as $k \to \infty$ from Eq. (1).

Then,

$$\begin{aligned}
n^{-1}\sum_{j=1}^{n} p_{S_j}^{-1} &= n^{-1}\sum_{k=-n}^{n} N_n(k)p_k^{-1} \\
&= n^{-1}\sum_{k=1}^{n}\left[N_n(k)p_k^{-1} + N_n(-k)p_{-k}^{-1}\right] + n^{-1}N_n(0)p_0^{-1} \\
&= n^{-1}\sum_{k=1}^{n}\left[N_n(k)(k\alpha_k - (k-1)\alpha_{k-1})\right. \\
&\quad \left. + N_n(-k)(k\beta_k - (k-1)\beta_{k-1})\right] + n^{-1}N_n(0)p_0^{-1} \\
&= n^{-1}\sum_{k=1}^{n-1} k\{(N_n(k) - N_n(k+1))\alpha_k \\
&\quad + [N_n(-k) - N_n(-k-1)]\beta_k\} \\
&\quad + N_n(n)\alpha_n + N_n(-n)\beta_n + n^{-1}N_n(0)p_0^{-1}
\end{aligned} \tag{7}$$

while $N_n(n) \overset{\text{a.s.}}{\to} 0$, $N_n(-n) \overset{\text{a.s.}}{\to} 0$ as $N \to \infty$, since

$$\sum_{n=1}^{\infty}\langle N_n(n)\rangle = \sum_{n=1}^{\infty}\langle N_n(-n)\rangle = \sum_{n=1}^{\infty} 2^{-n} = 1 \tag{8}$$

Furthermore, using I to denote the indicator function we have $N_n(0) = \sum_{k=0}^{n} I(S_k = 0)$ and $\sum_{n=1}^{\infty} n^{-1}I(S_n = 0) < \infty$ a.s. since $P(S_{2n+1} = 0) = 0$

and

$$P(S_{2n}=0)=\binom{2n}{n}\frac{1}{2^n}\sim(\pi n)^{-1/2}$$

as $n\to\infty$, so that $\sum_{n=1}^{\infty}n^{-1}P(S_n=0)<\infty$. It then follows from Kronecker's Lemma (e.g., Ref. 3, p. 31) that

$$n^{-1}N_n(0)=n^{-1}\sum_{k=0}^{n}I(S_k=0)\overset{\text{a.s.}}{\to}0 \tag{9}$$

as $n\to\infty$.

Now,

$$n^{-1}\sum_{k=1}^{n-1}k\{[N_n(k)-N_n(k+1)]\alpha_k+[N_n(-k)-N_n(-k-1)]\beta_k\}$$

$$=n^{-1}\sum_{k=1}^{n-1}k\{[N_n(k)-N_n(k+1)](\alpha_k-2\gamma)$$

$$+[N_n(-k)-N_n(-k-1)](\beta_k-2\gamma)\}$$

$$+2\gamma n^{-1}\sum_{k=-n}^{n}N_n(k)-2\gamma[N_n(n)+N_n(-n)+n^{-1}N_n(0)] \tag{10}$$

while

$$n^{-1}\sum_{k=-n}^{n}N_n(k)=1$$

Also, given $\epsilon>0$, we can choose $N=N(\epsilon)$ so large that $j^{\eta}|\alpha_j-2\gamma|<\epsilon$, $j^{\eta}|\beta_j-2\gamma|<\epsilon$ for $j\geqslant N$. Then, for $n>N$,

$$\Big|n^{-1}\sum_{k=N}^{n-1}k\{[N_n(k)-N_n(k+1)](\alpha_k-2\gamma)$$

$$+[N_n(-k)-N_n(-k-1)](\beta_k-2\gamma)\}\Big|$$

$$<\epsilon n^{-1}\sum_{k=N}^{n-1}k^{1-\eta}[|N_n(k)-N_n(k+1)|+|N_n(-k)-N_n(-k-1)|]$$

$$<2\epsilon n^{-1}\sup_k|N_n(k)-N_n(k+1)|\sum_{k=1}^{\max_{j\leqslant n}|S_j|}k^{1-\eta} \tag{11}$$

since $N_n(|k|)=0$ for $k>\max_{j\leqslant n}|S_j|$.

Next we need an a.s. bound for $\sup_k |N_n(k) - N_n(k+1)|$. To obtain this we first set $\tau = \min[k : k > 0, S_k = 0]$ and $\Delta = \sum_{k=1}^{\tau} I(S_k = 1)$, I denoting the indicator function. Then, $P(\Delta = 0) = \frac{1}{2}$ and for $k \geqslant 1$ the event $\{\Delta = k\}$ involves $S_1 = 1$, $k - 1$ returns to 1 from the right and then $S_\tau - S_{\tau-1} = -1$, so that $P(\Delta = k) = 2^{-(k+1)}$. It follows that $E\Delta = 1$ and $E\Delta^2 < \infty$. Furthermore, for fixed k we have

$$\Delta_1^{(k)} + \Delta_2^{(k)} + \cdots + \Delta_{N_n(k)-1}^{(k)} \leqslant N_n(k+1) \leqslant \Delta_1^{(k)} + \Delta_2^{(k)} + \cdots + \Delta_{N_n(k)}^{(k)}$$

where the $\Delta_i^{(k)}$ are independent and identically distributed, for fixed k and i running from 1 to ∞, with the distribution of Δ and are independent of $N_n(k)$, and hence

$$1 + \sum_{i=1}^{N_n(k)-1} \left(1 - \Delta_i^{(k)}\right) \geqslant N_n(k) - N_n(k+1) \geqslant \sum_{i=1}^{N_n(k)} \left(1 - \Delta_i^{(k)}\right)$$

Then, to show that

$$\sup_k |N_n(k) - N_n(k+1)| = o\left(n^{1/4+\delta}\right) \text{ a.s.} \tag{12}$$

for any $\delta > 0$ it suffices to establish that

$$\max_{|k| \leqslant (3n \log_2 n)^{1/2}} \left| \sum_{i=1}^{N_n(k)} \left(1 - \Delta_i^{(k)}\right) \right| = o\left(n^{1/4+\delta}\right) \text{ a.s.} \tag{13}$$

since

$$\overline{\lim_{n\to\infty}} (2n \log_2 n)^{-1/2} \sup_{j \leqslant n} |S_j| = 1 \text{ a.s.}$$

by the classical law of the iterated logarithm. But, for any $\epsilon > 0$ we have

$$\begin{aligned} A &= \sum_n P\left(\max_{|k| \leqslant (3n \log_2 n)^{1/2}} \left| \sum_{i=1}^{N_n(k)} \left(1 - \Delta_i^{(k)}\right) \right| > \epsilon n^{1/4+\delta} \right) \\ &\leqslant \sum_n \sum_{|k| \leqslant (3n \log_2 n)^{1/2}} P\left(\left| \sum_{i=1}^{N_n(k)} \left(1 - \Delta_i^{(k)}\right) \right| > \epsilon n^{1/4+\delta} \right) \\ &= \sum_n \sum_{|k| \leqslant (3n \log_2 n)^{1/2}} \sum_{j=1}^{\infty} P\left(\left| \sum_{i=1}^{j} \left(1 - \Delta_i^{(k)}\right) \right| > \epsilon n^{1/4+\delta} \right) P(N_n(k) = j) \end{aligned}$$

since $N_n(k)$ is independent of the $\Delta_i^{(k)}$. Furthermore, choosing an integer $r > 3/4\delta$ we have from Markov's inequality and the theorem of Ref. 4 that

$$P\left(\left| \sum_{i=1}^{j} \left(1 - \Delta_i^{(k)}\right) \right| > \epsilon n^{1/4+\delta} \right) \leqslant C_r \epsilon^{-2r} n^{-2r(1/4+\delta)} j^r$$

where C_r depends only on r, and hence

$$A \leqslant C_r \epsilon^{-2r} \sum_n \sum_{|k| \leqslant (3n \log_2 n)^{1/2}} n^{-2r(1/4+\delta)} \sum_{j=1}^{\infty} j^r P(N_n(k) = j)$$

$$= C_r \epsilon^{-2r} \sum_n n^{-2r(1/4+\delta)} \sum_{|k| \leqslant (3n \log_2 n)^{1/2}} \left\langle \left[N_n(k) \right]^r \right\rangle.$$

But, for $k \neq 0$, $\langle [N_n(k)]^r \rangle \leqslant \langle [N_n(0)]^r \rangle$ (since first passage to k must precede successive returns to k which in turn has the distribution of successive returns to the origin) so that

$$A \leqslant 2C_r \epsilon^{-2r} \sum_n n^{-2r(1/4+\delta)} (3n \log_2 n)^{1/2} \left\langle \left[N_n(0) \right]^r \right\rangle$$

Also,

$$\left\langle \left[N_n(0) \right]^r \right\rangle \sim \frac{2^{r/2} r!}{\Gamma(r/2+1)} n^{r/2} \qquad \text{as} \quad n \to \infty$$

using Eq. (10.29), p. 230 of Ref. 5 and this leads to $A < \infty$ (since $r > 3/4\delta$) from which Eq. (12) follows.

Finally, returning to Eq. (11) and using Eq. (12) we have

$$\Bigg| n^{-1} \sum_{k=1}^{n-1} k \left\{ \left[N_n(k) - N_n(k+1) \right] (\alpha_k - 2\gamma) + \left[N_n(-k) - N_n(-k-1) \right] (\beta_k - 2\gamma) \right\} \Bigg|$$

$$= o\left(n^{-3/4+\delta} \left(\max_{j \leqslant n} |S_j| \right)^{2-\eta} \right) = o(1) \text{ a.s.}$$

since $\max_{j \leqslant n} |S_j| = O(n \log_2 n)^{1/2}$ a.s. by the classical law of the iterated logarithm, while $\eta > 1/2$ and we can choose $0 < \delta < \frac{1}{2}(\eta - \frac{1}{2})$. This completes the proof of Theorem 2 in view of Eqs. (7)–(10). ■

Proof of Theorem 1. For fixed n let $\sigma_{k(n)}$ be defined by

$$\sigma_{k(n)} = \max\left[j : j \leqslant n, X_j \neq X_{j-1} \right]$$

so that $X_n = X_{\sigma_{k(n)}} = S_{k(n)}$, $\{S_j, j \geqslant 0\}$ being a simple random walk. Furthermore, using Theorem 2, $n^{-1} k(n) \to \gamma^{-1}$ a.s. as $n \to \infty$.

Now, using a result of Strassen (Ref. 6, Theorem 1.5, p. 320), and redefining the probability space as necessary, there is a standard Brownian motion $\{W(t), t \geqslant 0\}$ such that

$$S_j = W(j) + O\left(j^{1/4} (\log j \log_2 j)^{1/2} \right) \text{ a.s.}$$

as $j \to \infty$. Then, writing $k(n) = n\gamma^{-1}(1 + \epsilon_n)$, we have $\epsilon_n \to 0$ a.s. as $n \to \infty$

and

$$X_n = X_{\sigma_{k(n)}} = S_{k(n)} = W(k(n)) + O\left(\left[k(n)\right]^{1/4}\left[\log k(n)\log_2 k(n)\right]^{1/2}\right)$$
$$= W\left(n\gamma^{-1}(1+\epsilon_n)\right) + O\left(n^{1/4}(\log n \log_2 n)^{1/2}\right) \text{ a.s.}$$

which completes the proof since $\{\gamma^{1/2}W(t\gamma^{-1}),\ t \geqslant 0\}$ is also standard Brownian motion. ■

Proof of Corollary 1. Again write $k(n) = n\gamma^{-1}(1+\epsilon_n)$. From Theorem 1 we have

$$\gamma^{1/2}n^{-1/2}X_n = \gamma^{1/2}n^{-1/2}W(k(n)) + O\left(n^{-1/4}(\log n \log_2 n)^{1/2}\right) \text{ a.s.}$$

and, using $\overset{\mathrm{d}}{=}$ to denote equality in distribution,

$$n^{-1/2}W(k(n)) \overset{\mathrm{d}}{=} W\left(n^{-1}k(n)\right) \overset{\mathrm{d}}{\to} W(\gamma^{-1}) \overset{\mathrm{d}}{=} \gamma^{-1/2}N(0,1)$$

since $n^{-1}k(n) \to \gamma^{-1}$ a.s. $n \to \infty$. The result (i) then follows immediately.

To establish (ii) we first note that, again using Theorem 1,

$$\overline{\lim_{n\to\infty}} \frac{\gamma^{1/2}X_n}{(2n\log_2 n)^{1/2}} = \overline{\lim_{n\to\infty}} \frac{W(k(n))}{\left[2k(n)\log_2 k(n)\right]^{1/2}} \left[\frac{\gamma k(n)\log_2 k(n)}{n\log_2 n}\right]^{1/2}$$

and

$$\overline{\lim_{n\to\infty}} \frac{W(k(n))}{\left[2k(n)\log_2 k(n)\right]^{1/2}} \leqslant 1 \text{ a.s.}$$

from the law of the iterated logarithm for Brownian motion, while

$$\overline{\lim_{n\to\infty}} \frac{\gamma k(n)\log_2 k(n)}{n\log_2 n} = 1 \text{ a.s.}$$

since $n^{-1}k(n) \to \gamma^{-1}$ a.s. as $n \to \infty$. Consequently,

$$\overline{\lim_{n\to\infty}} \frac{\gamma^{1/2}X_n}{(2n\log_2 n)^{1/2}} \leqslant 1 \tag{14}$$

On the other hand, recalling that $\{X_{\sigma_k} = S_k,\ k \geqslant 1\}$ is simple random walk, we have

$$\overline{\lim_{k\to\infty}} \frac{\gamma^{1/2}X_{\sigma_k}}{(2\sigma_k\log_2\sigma_k)^{1/2}} = \overline{\lim_{k\to\infty}} \frac{S_k}{(2k\log_2 k)^{1/2}} \left(\frac{\gamma k\log_2 k}{\sigma_k\log_2\sigma_k}\right)^{1/2} = 1 \text{ a.s.} \tag{15}$$

since

$$\overline{\lim_{k\to\infty}} (2k\log_2 k)^{-1/2}S_k = 1 \text{ a.s.}$$

by the ordinary law of the iterated logarithm, while $k^{-1}\sigma_k \to \gamma$ by Theorem 2. It follows from Eqs. (14) and (15) that

$$\overline{\lim_{n\to\infty}} \frac{\gamma^{1/2} X_n}{(2n \log_2 n)^{1/2}} = 1 \text{ a.s.}$$

and similar considerations apply to the corresponding $\underline{\lim}$.

Proof of Corollary 2. It suffices to show that $\{n^{-1}X_n^2,\ n \geqslant 1\}$ is uniformly integrable for then the result follows immediately from part (i) of Corollary 1 using, e.g., Ref. 7, Theorem 5.4, p. 32. However, we note that $\{X_n,\ n \geqslant 1\}$ is a martingale and hence, using the theorem of Ref. 4,

$$\langle X_n^4 \rangle \leqslant Cn^2$$

for some finite constant C. The uniform integrability of $\{n^{-1}X_n^2,\ n \geqslant 1\}$ then follows since (using I for the indicator function)

$$n^{-1}\left\langle X_n^2 I(|X_n| > \lambda n^{1/2})\right\rangle \leqslant n^{-2}\lambda^{-2}\left\langle X_n^4 \right\rangle \leqslant C\lambda^{-2}$$

and this completes the proof.

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